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# Computing the Brauer Group of Graded Azumaya Algebras from Its Subgroups*

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Let $R$ be a connected commutative ring with unity and $G$ a finite abelian group of order $n$ and exponent $m$. Assume that $n$ is a unit in $R$, $\mathrm{Pic}_m(R)=0$ and $R$ contains a primitive $m$th root of unity.

In [9], Long defined $BD(R, G)$, a generalized Brauer group of $G$-dimodule algebras, i.e., algebras with a $G$-grading upon which $G$ acts as a group of grade-preserving automorphisms. This is a generalization of the Brauer–Wall group [17, 15], and the graded Brauer groups of Knus [8] and Childs, Garfinkel, and Orzech [4]. Just as Clifford algebras are elements of the Brauer–Wall group, so the generalized Clifford algebras of [12, 13, 16] are elements of Long's generalized Brauer group.

$BD(R, G)$ was first computed for $R$ a separably closed field and $G$ cyclic of order a product of primes; later $BD(R, G)$ was computed for any cyclic $G$ and more general $R$, [9, 14, 2, 3]. The key to these computations was the normal subgroup of central $G$-Azumaya algebras [2]. However, as Orzech pointed out in [14], if $G$ is noncyclic, the set of central $G$-Azumaya algebras may not be a subgroup. Therefore, to compute $BD(R, G)$ for noncyclic $G$, one must try a new approach.

One approach is that taken by Childs [5] in which he studies a group of graded Galois extensions, $\mathrm{Galz}_\phi(R, G)$, defined to provide an image of $B_\phi(R, G)$ under a map $\pi$ with kernel the usual Brauer group of $R$. If $G\cong G^*$, then $BD(R, G)\cong B_\phi(R, G\times G)$ for a particular bilinear map $\phi$. He finds that for $G$ a direct product of cyclic groups of order $p^e$, $BD(R, G)$ is described by the exact sequences

$$0\to B(R)\to BD(R, G)\to \mathrm{Galz}_\phi(R, G\times G)\to 0$$

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and

$$0 \to \mathrm{Comm}(R, G \times G) \to \mathrm{Galz}_{\phi}(R, G \times G) \to \mathcal{O} \to 0,$$

where $B(R)$ is the usual Brauer group of $R$, $\mathrm{Comm}(R, G \times G)$ is the group of commutative Galois extensions of $R$ with group $G \times G$ and $\mathcal{O}$ is an orthogonal group of matrices. The computation of $BD(R, G)$ then depends upon the computation of $\mathcal{O}$, and, as Childs points out, direct methods of computation become quite involved.

Another approach is to try to describe the structure of $BD(R, G)$ via its subgroups. In [7], Deegan defined and described a subset $BT(R, G)$ of $BD(R, G)$ which is a subgroup isomorphic to $\mathrm{Aut}(G)$ for all $G$. In Section 1 of this paper we introduce a subgroup $\mathscr{A}$ of central algebras with trivial action cocycle and show that $\mathscr{A} \cong BC(R, G) \times BT(R, G)$. We also denote by $\mathscr{G}$ the subgroup generated by $BM(R, G)$ and $BT(R, G)$. $\mathscr{A}$ and $\mathscr{G}$ then generate a subgroup $B(R, G)$ which contains all central algebras. (If $G$ is cyclic, $B(R, G)$ is the subgroup of central $G$-Azumaya algebras.)

In Section 2 we demonstrate an example of a Brauer group $BD(R, G)$ with noncyclic $G$ by computing $BD(K, C_2 \times C_2)$ for $K$ a separably closed field. $BD(K, C_2 \times C_2)$ is a nonabelian group of order 72 with a normal subgroup isomorphic to $S_3 \times S_3$. The computation is done by studying the subgroups $B(K, C_2 \times C_2)$, $\mathscr{A}$ and $\mathscr{G}$, of $BD(K, C_2 \times C_2)$.

The final section offers some conjectures on the general structure of $BD(R, G)$.

## Preliminaries

We review a few pertinent definitions briefly here. Let $R$ be a commutative ring with unit and $G$ a finite abelian group. All algebras and modules are $R$-algebras and $R$-modules. We write $A \otimes B$ for $A \otimes_R B$, $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_R(A, B)$, etc. Formulae defined only for the homogeneous elements of a graded module are to be extended by linearity. $U(R)$ is the group of units of $R$.

Definitions of $G$-modules, $G$-comodules, $G$-dimodules, $G$-module algebras, smash product, etc., may be found in [9]. A $G$-dimodule algebra $A$ is called $G$-Azumaya if $A$ is an $R$-progenerator and the $G$-dimodule algebra maps $F$ and $G$ described in [9, p. 241] are isomorphisms. If $A$ and $B$ are $G$-Azumaya, so are $\bar{A}$, the $G$-opposite algebra of $A$, and $A \# B$, the smash product of $A$ and $B$. If $M$ is an $R$-progenerator $G$-dimodule, $\mathrm{End}(M)$ is $G$-Azumaya. Note that $G$-Azumaya algebras are separable [14, Proposition 2.2]. $BD(R, G)$ is defined to be the group of equivalence classes of $G$-Azumaya algebras with the equivalence relation $\sim$ being given by $A \sim B$ if there exist $G$-dimodule $R$-progenerators $M$ and $N$ such that $A \# \mathrm{End}(M) \cong B \# \mathrm{End}(N)$. The equivalence class of $A$ is written $[A]$; multiplication is given by the smash product. Algebras equivalent in $BD(R, G)$ have isomorphic centres. Recall that $A \# \mathrm{End}(M) \cong A \otimes \mathrm{End}(M) \cong \mathrm{End}(M) \otimes A \cong \mathrm{End}(M) \# A$.

$B(R)$, the usual Brauer group of $R$, is embedded in $BD(R, G)$ by giving each element of $B(R)$ trivial $G$-grading and action. $BM(R, G)$ denotes the subgroup of classes for which a representative has trivial $G$-grading; similarly $BC(R, G)$ is the subgroup of classes for which a representative has trivial action. $BM(R, G) \cap BC(R, G)$ is $B(R)$.

If $G$ has order $n$ and exponent $m$, we assume that $1/n \in R$, $\mathrm{Pic}_m(R) = 0$ and $R$ contains a primitive $m$th root of unity. Let $A$ be a fully graded $G$-Azumaya algebra (i.e., $A_\sigma A_\tau = A_{\sigma\tau}$ for all $\sigma$, $\tau$ in $G$) with centre $Z$. Let $H$ be the subgroup of gradings of $Z$ and let $I$ be the subgroup of those elements of $G$ which leave $Z$ elementwise fixed. Then by [3], $H$ is a direct summand of $G$. Since $Z$ is a Galois $(G/I)$ $R$-object and a Galois $RH$-object [14], then $A \cong A^{R(G/I)} \otimes Z \cong Z \otimes A^{HR}$ as $G$-dimodule algebras (if there is a well-defined $G/I$ action on $A$.) If $H = G$ then $A \cong A^G \# Z \cong Z \# A_1$.

If $f \in Z^2(G, U(R))$ and $A$ is a $G$-dimodule algebra, then $A_f$ will denote the $G$-dimodule $A$ with new multiplication given by $a \cdot b = f(\alpha, \beta)\, ab$ for $a \in A_\alpha$, $b \in B_\beta$. If $f$ and $g$ are cohomologous cocycles, then $A_f \cong A_g$, so that $f$ may be chosen from $H^2(G, U(R))$. If $\phi$ is a bilinear map from $G \times G$ to $U(R)$, then $RG_f^\phi$ denotes the group ring $RG$ with the usual $G$-grading, multiplication induced by $f$ and $G$-action given by $\sigma(x) = \phi(\sigma, \tau)\, x$ for $x \in RG_\tau$.

Unless otherwise specified, the letters $M$ and $N$ will denote finitely generated projective $G$-dimodules and $\mathrm{End}(M)$, $\mathrm{End}(N)$ will have the usual $G$-Azumaya algebra structure.

## 1. Subgroups of $BD(R, G)$

First, let us review the group $BT(R, G)$ defined and described by Deegan in [7]. For a central separable $G$-module algebra $A$, each element of $G$ acts innerly, so that, for $\sigma \in G$, there is an $x_\sigma \in U(A)$ such that $\sigma(a) = x_\sigma a x_\sigma^{-1}$ for all $a \in A$. The "action conjugates" $x_\sigma$ give rise to an element $f$ of $H^2(G, U(R))$, called the action cocycle of $A$, given by $f(\sigma, \tau) = x_\sigma x_\tau x_{\sigma\tau}^{-1}$. Similarly, under the given conditions, the $G$-grading (or $GR$-action) on $A$ can be viewed as a separate $G$-action on $A$, so that this action too is inner and gives rise to a set of "grading conjugates" and a grading cocycle. The action and grading cocycles are the same for each $A \in [A]$. By [7, Proposition 2.13], $BT(R, G)$ is the subgroup of $BD(R, G)$ defined by the following property: A central Brauer class of the form $[\mathrm{End}(M)]$ lies in $BT(R, G)$ if and only if both the action and grading cocycles are trivial in

$H^2(G, U(R))$. The map $\beta$ defined in [2, Sect. 1] is an isomorphism on $BT(R, G)$, so that $BT(R, G) \cong \mathrm{Aut}(G)$ [7, Theorem 3.2].

Now we define the subgroup $\mathscr{A}$ of $BD(R, G)$ of Brauer classes with trivial action cocycle.

DEFINITION 1.1. Let $\mathscr{A} = \{[A] \in BD(R, G);\ [A]$ is central and has trivial action cocycle$\}$.

PROPOSITION 1.2. *$\mathscr{A}$ is a subgroup of $BD(R, G)$ and*

$$1 \to BC(R, G) \to \mathscr{A} \xrightarrow{\beta} \mathrm{Aut}(G) \to 1$$

*is a split short exact sequence.*

*Proof.* Let $A \in [A] \in \mathscr{A}$ with set of action conjugates $x_\gamma$. Since $A$ has trivial action cocycle, the $x_\gamma$ commute and are invariant under the action of $G$. Thus, for $B \in [B] \in \mathscr{A}$, $A \# B \cong A \otimes B$ as $G$-module algebras under the isomorphism $a \# b \to \sum_{\gamma \in G} a x_\gamma \otimes b_\gamma$ (cf. [2, p. 519]). If $y_\delta$ is a set of action conjugates for $B$, then $x_\alpha \otimes y_\alpha$ is a set of action conjugates for $A \otimes B$. Thus the action cocycle for $A \otimes B$ (and $A \# B$) is trivial, and $A \otimes B$ (and $A \# B$) are central. The above argument may be applied to show that $A \# \bar{A} \cong A \otimes \bar{A}$ so that $\bar{A}$ is central. Since $A \otimes \bar{A}$ and $A$ have trivial action cocycles, $\bar{A}$ must also.

Therefore $\mathscr{A}$ is a subgroup of $BD(R, G)$ containing $BT(R, G)$, and it is easy to check that the map $\beta$ defined in [2, p. 519] is well-defined on $\mathscr{A}$, giving the short exact sequence

$$1 \to \mathrm{Ker}\, \beta \to \mathscr{A} \xrightarrow{\beta} \mathrm{Aut}(G) \to 1.$$

Since $\beta$ is an isomorphism from $BT(R, G)$ to $\mathrm{Aut}(G)$, the sequence splits.

Now we must show that $\mathrm{Ker}\, \beta = BC(R, G)$. Clearly $BC(R, G) \subset \mathrm{Ker}\, \beta$. Let $A \in [A] \in \mathrm{Ker}\, \beta$ with set of action conjugates $\{x_\sigma : \sigma \in G\}$. The argument is then essentially that of Theorem 1.2 [2, p. 522]. Let $B$ be the $G$-graded algebra $A$ but with trivial $G$-action; $[B] \in BC(R, G)$. Let $A'$ be the $G$-graded module $A$ but with $G$-action given by $\sigma(a') = (x_\sigma a)'$. Then $\bar{B} \# A = B^0 \otimes A \cong A \otimes B^0 \cong \mathrm{End}(A')$ as $G$-dimodule algebras. Therefore $[\bar{B} \# A]$ is trivial in $BD(R, G)$ and $[A] \in BC(R, G)$.

Note that since the sequence in Proposition 1.2 splits, the elements of $BT(R, G)$ and $BC(R, G)$ commute.

We would like to obtain a similar subgroup of Brauer classes with trivial grading cocycle. If $R$ is a separably closed field, the definition below of $\mathscr{G}$ yields a subgroup containing all Brauer classes with trivial grading cocycle; in the example in Section 2, every Brauer class in $\mathscr{G}$ has trivial grading cocycle.

DEFINITION 1.3. Let $\mathscr{G}$ be the subgroup of $BD(R, G)$ generated by $BM(R, G)$ and $BT(R, G)$.

The generators of $\mathscr{G}$ have trivial grading cocycle. To show that if $R$ is a separably closed field, $\mathscr{G}$ contains every central Brauer class with trivial grading cocycle, we need

LEMMA 1.4. *Let $f \in H^2(G, U(R))$. Then there is an $[X] \in BM(R, G)$ with action cocycle $f$.*

*Proof.* Recall that by [1], $BM(R, G) \cong B(R) \times T$ where $T$ is a subgroup of $BM(R, G)$ isomorphic to $\mathrm{Gal}(R, RG)$. Now define a map $\alpha: T \to H^2(G, U(R))$ by mapping $[W]$ to its action cocycle. Since for $[W]$, $[Z] \in T$, $W \# Z = W \otimes Z$, $\alpha$ is a group homomorphism. By examining the proof of Theorem 1.2 [2], we see that $\alpha = j \circ \rho \circ i$ where $\rho$ and $i$ are as in Theorem 1.2 [2] and $j(f) = g$ with $g(\sigma, \tau) = f(\sigma^{-1}, \tau^{-1})$. Since $j$ and $\rho \circ i$ are isomorphisms, so is $\alpha$, and there must exist $[X] \in T \subset BM(R, G)$ with action cocycle $f$.

PROPOSITION 1.5. *Suppose $R$ is a separably closed field. Then $\mathscr{G}$ contains every central Brauer class with trivial grading cocycle.*

*Proof.* Let $A$ be a central $G$-Azumaya algebra with trivial grading cocycle. If $A$ also has trivial action cocycle, then $[A] \in BT(R, G)$. Suppose then that $A$ has action cocycle $f$ and choose $[X] \in BM(R, G)$ with action cocycle $f^{-1}$. (Such an $[X]$ exists by Lemma 1.4.) Then $A \# X = A \otimes X$ has trivial action cocycle, so that $[A][X] \in BT(R, G)$ and thus $[A] \in \mathscr{G}$.

Now we define the subgroup $B(R, G)$.

DEFINITION 1.6. Let $B(R, G)$ be the subgroup of $BD(R, G)$ generated by $\mathscr{A}$ and $\mathscr{G}$, i.e., by $BC(R, G)$, $BM(R, G)$, and $BT(R, G)$.

If every cocycle in $H^2(G, U(R))$ is abelian, then $B(R, G)$ is the subgroup of central algebras of $BD(R, G)$ (cf. [2, Theorem 1.2]), and in any case $B(R, G)$ contains all the central algebras. It is shown in [3] that $B(R, G)$ is normal of index 2 in $BD(R, G)$ for $G$ cyclic. This is also true if $R$ is a separably closed field and $G = C_p \times C_p$, $p$ a prime.

PROPOSITION 1.7. *$B(R, G)$ contains every central Brauer class.*

*Proof.* Let $A$ be a central $G$-Azumaya algebra. If $A$ has trivial action cocycle, then $[A] \in \mathscr{A} \subset B(R, G)$. If $A$ has action cocycle $f$, choose $[X] \in BM(R, G)$ with action cocycle $f^{-1}$ and use the same argument as in Proposition 1.5.

THEOREM 1.8. *Suppose $R$ is a separably closed field and $G \cong C_p \times C_p$ where $C_p$ is the cyclic group of order $p$, $p$ prime. Then $B(R, G)$ is the subgroup of $G$-Azumaya algebras whose rank is $m^2$ for some integer $m$ and thus $B(R, G)$ is normal of index 2.*

*Proof.* Since $R$ is separably closed, every central separable algebra has rank $m^2$ for some integer $m$. Thus, since $B(R, G)$ is generated by central algebras, every Brauer class in $B(R, G)$ has rank a square.

Conversely, suppose $A$ is a $G$-Azumaya algebra whose rank is a square. If $A$ is central, then $[A] \in B(R, G)$ by Proposition 1.7. If $A$ has centre $Z$, $Z \neq R$, then let $H$ denote the group of gradings of $Z$. $H$ is either all of $G$ or is isomorphic to $C_p$.

Suppose $H \cong C_p$. By [14, Proposition 2.11], $Z \cong RH_f^{\phi}$, and $Z$ is a Galois $RH$-object. Thus $A \cong Z \otimes A^{HR}$ as $G$-dimodule algebras, and $A^{HR}$ is a central separable algebra, and therefore of rank $m^2$. Then the rank of $A$ is $pm^2$ which is not a square.

Therefore $H = G$, and $A \cong Z \# A_1$ with $A_1$ in $BM(R, G)$. To show $A \in B(R, G)$, we must now show that $Z \in B(R, G)$. $Z \cong RG_f^{\phi}$ with $\phi$ a nondegenerate bilinear map from $G \times G$ to $U(R)$ and $f$ abelian and therefore trivial under the given conditions. By [14, p. 540], $\bar{Z} \sim RG_{\phi}^{\phi}$. If $\phi$ is nonabelian, then it is easy to check that $RG_{\phi}^{\phi}$ is central. Then $RG_{\phi}^{\phi}$, and therefore its inverse, $Z$, must lie in $B(R, G)$. If $\phi$ is abelian, then $Z = \bar{Z}$. Choose a nonabelian nondegenerate bilinear map $\chi$ from $G \times G$ to $U(R)$. By [14], $RG^{\chi}$ is $G$-Azumaya and, by the previous argument, $RG^{\chi}$ lies in $B(R, G)$. Again, an easy computation shows that $RG^{\chi} \# RG^{\phi}$ is central, so that $RG^{\phi}$ is also in $B(R, G)$.

## 2. COMPUTATION OF $BD(K, C_2 \times C_2)$

In this section $K$ will denote a separably closed field with characteristic different from 2. Let $G = C_2 \times C_2 = \langle \sigma \rangle \times \langle \tau \rangle$, the Klein 4-group. Recall that $H^2(G, K^*) \cong C_2$ so that here a cocycle $f$ is trivial if and only if $f$ is abelian. $B(K)$ is trivial and by [9, Theorem 2.7], $BD(K, C_2) \cong C_2$. By [1, Theorem 1.4], $BM(K, G) \cong BC(K, G) \cong H^2(G, K^*) \cong C_2$. Let $[X]$ and $[Y]$ generate $BM(K, G)$ and $BC(K, G)$, respectively, with $X \in [X]$ having trivial $G$-grading and $Y \in [Y]$ having trivial $G$-action. A suitable $Y$ would be $KG_f$ with trivial $G$-action, the usual $G$-grading on the group ring and $f$ the nontrivial element of $H^2(G, K^*)$.

$BT(K, G) \cong \mathrm{Aut}(G) \cong S_3$, the symmetric group on three elements. Following [7], we write $BT(K, G) = \{[B_i], 1 \leqslant i \leqslant 6\}$ with $B_i \in [B_i]$, $[B_1]$ the identity, $[B_2]$, $[B_3]$, $[B_4]$ the elements of order 2 and $[B_2][B_3] = [B_6]$, $[B_3][B_2] = [B_5]$.

Now let us label all nondegenerate bilinear maps from $G \times G$ to $K^*$ (extend formulae by bilinearity) by writing

$$\phi: \quad \begin{array}{c|c} a & b \\ \hline c & d \end{array} \quad \text{to denote } \phi\colon G \times G \to K^* \text{ with}$$

$$\phi(\sigma, \sigma) = a, \quad \phi(\sigma, \tau) = b,$$

$$\phi(\tau, \sigma) = c, \quad \phi(\tau, \tau) = d.$$

$$\phi_1: \begin{array}{c|c} +1 & -1 \\ \hline -1 & +1 \end{array}, \qquad \phi_2: \begin{array}{c|c} +1 & -1 \\ \hline -1 & -1 \end{array}, \qquad \phi_3: \begin{array}{c|c} -1 & -1 \\ \hline -1 & +1 \end{array},$$

$$\phi_4: \begin{array}{c|c} -1 & +1 \\ \hline +1 & -1 \end{array}, \qquad \phi_5: \begin{array}{c|c} -1 & +1 \\ \hline -1 & -1 \end{array}, \qquad \phi_6: \begin{array}{c|c} -1 & -1 \\ \hline +1 & -1 \end{array}.$$

Denote $KG^{\phi_i}$ by $KG(i)$. The algebras $KG(i)$, $i = 1, \ldots, 6$, are $G$-Azumaya [14, Proposition 2.8]. $KG(i)$ has order 2 for $i = 1, 2, 3$ or 4; since $\phi_5$ and $\phi_6$ are nonabelian, $KG(5)$ and $KG(6)$ have central inverses.

Example 4.3 of [7] shows that $B_2 \sim KG(1) \# KG(2)$ and $B_3 \sim KG(1) \# KG(3)$. A similar computation shows that $B_4 \sim KG(1) \# KG(4)$, $B_5 \sim B_3 \# B_2 \sim \bar{B}_3 \# B_2 \sim KG(3) \# KG(2)$ and $B_6 \sim \bar{B}_2 \# B_3 \sim KG(2) \# KG(3)$.

We now compute $BD(K, G)$ by first describing $B(K, G)$ and then noting that, by Theorem 1.8, $B(K, G)$ is a normal subgroup of index 2 of $BD(K, G)$. Recall that if all cocycles in $H^2(G, U(R))$ are abelian, then $B(R, G)/B(R) \cong BM(R, G)/B(R) \times BC(R, G)/B(R) \times BT(R, G)$ by [2, Theorem 1.7] and [7, Theorem 3.2]. If the cocycles in $H^2(G, U(R))$ are not all abelian, then the elements of $BM(R, G)$ and $BC(R, G)$ may not commute. However, it turns out that in this example, $B(K, G)$ is still the direct product of $BT(K, G)$ and the subgroup generated by $BM(K, G)$ and $BC(K, G)$.

Proposition 2.1. *The subgroup $S$ of $B(K, G)$ generated by $BM(K, G)$ and $BC(K, G)$ is isomorphic to $S_3$.*

*Proof.* Since $S$ is generated by $[X]$ and $[Y]$, we check how $[X]$ and $[Y]$ multiply. $Y \# X \cong Y \otimes X$, a central algebra. Recall that $Y = KG_f$ where $f$ is nontrivial in $H^2(G, K^*)$ and $KG_f$ has the usual $G$-grading on the group ring and trivial $G$-action. By the usual argument (see Section 1 or [2, p. 519]), $X \# Y \cong X \otimes Y_f$ as $G$-graded algebras. But $Y_f = (KG_f)_f = KG$ with the usual $G$-grading and multiplication. Therefore the centre $Z$ of $X \# Y$ has group of gradings $G$. Then $X \# Y \cong (X \# Y)^G \# Z \cong X^G \# Y \# Z \cong Y \# Z$ and $X \# Y \cong Z \# (X \# Y)_1 = Z \# (X \# Y_1) \cong Z \# X$. Thus $[Z] = [X \# Y \# X] = [Y \# X \# Y]$, an element of order 2 in $BD(K, G)$, so that $[X \# Y]^3 = 1$ in $BD(K, G)$. Since $Z$ is commutative, $Z \sim KG(i)$ for some $i$, $1 \leqslant i \leqslant 4$. It is now straightforward to check that $S=\{[1], [X], [Y], [X\# Y\# X]=[Y\# X\# Y], [X\# Y]=[Y\# X]^2, [X\# Y]^2=[Y\# X]\}$ has the structure of $S_3$.

We now show that $B(K, G)$ is isomorphic to $S_3\times S_3$ by a series of lemmas. $X$, $Y$ and $B_i$, $i\in\{1,..., 6\}$ are as above.

Lemma 2.2. *$X\# B_i$ is a central algebra for all $i\in\{1,..., 6\}$.*

*Proof.* As in the proof of Proposition 2.1, $X\# B_i\cong X\otimes(B_i)_f$ as $R$-algebras. We show that $(B_i)_f$ is central. Note that since $f(1, \gamma)=f(\gamma, 1)=1$ for all $\gamma\in G$, multiplication with elements of $(B_i)_1$ is the same in $B_i$ and $(B_i)_f$.

Suppose $B_i\sim KG(j)\# KG(k)$ and let $e_\alpha, \alpha\in G$, and $h_\beta$, $\beta\in G$, be free bases for $KG(j)$ and $KG(k)$. We show that $w=e_\alpha\# h_\beta$ is not central unless $\alpha=\beta=1$. Suppose $\alpha\neq\beta$, so that $\alpha\beta\neq 1$. Then since $\phi_j$ is nondegenerate, there exists $\gamma\in G$ such that $\phi_j(\alpha\beta, \gamma)\neq 1$. Let $z=e_\gamma\# h_\gamma\in(B_i)_1$. Then $wz=zw$ if and only if $\phi_j(\alpha, \gamma)=\phi_j(\beta, \gamma)=\phi_j^{-1}(\beta, \gamma)$, i.e., if and only if $\phi_j(\alpha\beta, \gamma)=1$. If $\alpha=\beta$, choose $\gamma$ such that $\phi_j(\gamma, \alpha)\neq 1$ and then $1\# h_\gamma$ will not commute with $w$.

Lemma 2.3. *Every Brauer class in $\mathscr{G}$, the subgroup generated by $[X]$ and $BT(K, G)$ is central. Therefore $\mathscr{G}$ is a proper subgroup of $B(K, G)$.*

*Proof.* The second statement follows from the first since $B(K, G)$ contains noncentral Brauer classes, for example $KG(5)$ and $KG(6)$.

Let $[C]\in\mathscr{G}$. Then $C\sim B_{i_1}\# X\#\cdots\# B_{i_n}$ with $i_j\in\{1,..., 6\}$. Let $m$ be the number of nontrivial $B_i$ which occur in the product; we use induction on $m$. If $m=1$, then $C$ is equivalent to an algebra of the form $X$, $B_i$, $X\# B_i$ or $B_i\# X$, all of which are central. Suppose that every product containing $m-1$ nontrivial $B_i$ is central; let $C$ be such a product. We may assume that $C$ has $X$ as rightmost factor; if not then replace $C$ by $C\# X=C\otimes X$ with the same centre. If $C$ has trivial action cocycle, then $C\# B_j\cong C\otimes B_j$ as $G$-module algebras; otherwise $C\# B_j\cong C\otimes(B_j)_f$ as $R$-algebras. In both cases $C\# B_j$ is central.

Lemma 2.4. *$\mathscr{G}$ is the set of Brauer classes with trivial grading cocycle.*

*Proof.* By Proposition 1.5, $\mathscr{G}$ contains every central Brauer class with trivial grading cocycle.

Every Brauer class of the form $[B_j\# X]=[B_j\otimes X]$ has trivial grading cocycle since $[B_j]$ and $[X]$ do. We show that every Brauer class in $\mathscr{G}$ has the form $[X]$, $[B_i]$ or $[B_j\# X]$.

By Lemma 2.2, $X\# B_i$ is central. If $X\# B_i$ has trivial action cocycle, then $[X \# B_i] \in \mathscr{A}$, which implies that $[X] \in \mathscr{A}$, a contradiction. Therefore $X \# B_i$ has action cocycle $f$ and $X \# B_i \# X = (X \# B_i) \otimes X$ has trivial action cocycle. Thus $[X \# B_i \# X] \in \mathscr{A}$ and $X \# B_i \# X \sim B_j$ or $X \# B_i \# X \sim Y \# B_k$.

If $X \# B_i \# X \sim Y \# B_k$ then $[Y] = [X \# B_i \# X \# \bar{B}_k] \in \mathscr{G}$, and $\mathscr{G} = B(K, G)$. This is impossible by Lemma 2.3. Therefore $X \# B_i \# X \sim B_j$ and $X \# B_i \sim B_j \# X$.

LEMMA 2.5. *There is a split short exact sequence*

$$1 \to BT(K, G) \to \mathscr{G} \to BM(K, G) \to 1.$$

*Therefore* $\mathscr{G} \cong BT(K, G) \times BM(K, G) \cong S_3 \times C_2$ *and* $[X]$ *commutes with* $[B_i]$, $i \in \{1,..., 6\}$.

*Proof.* Map $\mathscr{G}$ onto $BM(K, G)$ by mapping all Brauer classes with nontrivial action cocycle to $[X]$ and those with trivial action cocycle to the identity. From Lemma 2.4, every Brauer class with nontrivial action cocycle has the form $[B_i \# X]$, $i \in \{1,..., 6\}$, and since $[B_i \# X][B_j \# X] \sim [B_i \# X][X \# B_k] \in BT(K, G)$, our map is a group homomorphism.

From Proposition 1.2 and Lemma 2.5, the elements of $S$ (generated by $[X]$ and $[Y]$) commute with those of $BT(K, G)$. Also $S \cap BT(K, G)$ is trivial. For clearly neither $[X]$ nor $[Y]$ lies in $BT(K, G)$, and, since $[X \# Y]$ and $[X \# Y \# X]$ are not central, $[X \# Y]$, $[X \# Y \# X]$ and $[Y \# X]$ do not lie in $BT(K, G)$. Summarizing the above discussion, we obtain the following theorem.

THEOREM 2.6. $BD(K, C_2 \times C_2)$ is a group of order 72 containing a normal subgroup $B(K, C_2 \times C_2)$ of order 36 isomorphic to $S_3 \times S_3$, where $S_3$ is the symmetric group on three elements.

*Remark* 2.7. The exact sequence in Theorem 2.6 does not split. For if it did, all Brauer classes with $C_2$-graded centre would commute with the elements of $B(K, G)$. We give a counterexample.

Let $H_1 = \{1, \sigma\}$, $H_2 = \{1, \tau\}$ and $H_3 = \{1, \sigma\tau\}$ be the subgroups of $G$ isomorphic to $C_2$. Let $\chi_1: G \times H_1 \to K^*$ be the bilinear map defined by $\chi_1(\sigma, \sigma) = \chi_1(\tau, \sigma) = -1$, let $\chi_2: G \times H_2 \to K^*$ be the bilinear map defined by $\chi_2(\sigma, \tau) = +1 = -\chi_2(\tau, \tau)$ and let $\chi_3: G \times H_3 \to K^*$ be the bilinear map defined by $\chi_3(\sigma, \sigma\tau) = -1 = -\chi_3(\tau, \sigma\tau)$.

It is easy to verify that the $KH_i^{\chi_i}$ are nonequivalent $G$-Azumaya algebras of order 2. A computation using the table in [7, p. 237] shows that $KH_1^{\chi_1} \# KH_3^{\chi_3} \sim KG(5) \sim KH_2^{\chi_2} \# KH_1^{\chi_1}$. Thus $KH_1^{\chi_1} \# KG(5)$ is not equivalent to $KG(5) \# KH_1^{\chi_1}$.

## 3. The General Structure of $BD(R, G)$

The computation of $BD(K, C_2 \times C_2)$ in Section 2 suggests that the following may be the general structure of $BD(R, G)$.

(i) $B(R, G)$, the smallest subgroup of $BD(R, G)$ containing all the central Brauer classes, is either $BD(R, G)$ or is normal of index 2.

(ii) $B(R, G)$ is the direct product of $BT(R, G)$ and $S$, the subgroup generated by $BM(R, G)$ and $BC(R, G)$.

The example in Section 2 shows one other interesting feature; $BD(K, C_2 \times C_2)$ is generated by $[X]$, $[Y]$ and the images of $BD(K, C_2)$ in $BD(K, C_2 \times C_2)$. The proof uses the same type of technique as Remark 2.7. It would be interesting to know, for example, if $BD(K, C_2 \times C_2 \times C_2)$ is generated by $BM(K, C_2 \times C_2 \times C_2)$, $BC(K, C_2 \times C_2 \times C_2)$ and the embeddings of $BD(K, C_2)$ in $BD(K, C_2 \times C_2 \times C_2)$.

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